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Stirling number identities: interconsistency of q-analogues

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Abstract. q-analogues of Stirling number identities are formulated, and the interconsistency among the q-analogues of the Stirling numbers and of the binomial coefficients is investigated. The close relation with the normal ordering problem for Arik–Coon-type q-bosons plays a central role in the derivations presented.

1. Introduction

The advent of quantum groups has drawn wide attention to various facets of q-algebra and q-analysis. Some of the most elementary and well known results relate to the properties of q-integers, the simplest variant of which is defined as

$$[k]_q = \frac{q^k - 1}{q - 1}.$$

These q-integers satisfy the q-arithmetic relation

$$[k]_q + q^k[\ell]_q = [k+\ell]_q = [\ell]_q + q^\ell[k]_q \tag{1}$$

that will be needed later. They give rise to the q-factorial

$$[k]_q! = [k-1]_q! \cdot [k]_q \qquad [0]_q! = 1$$

and to the q-exponential

$$\exp_q(x) = \sum_{k=0}^{\infty} \frac{x^k}{[k]_q!}.$$

In this paper we pay particular attention to some of the properties associated with Arik–Coon q-boson operators [1], that satisfy

$$[a, a^\dagger]_q \equiv aa^\dagger - qa^\dagger a = 1.$$

In the corresponding Fock space

$$a^\dagger|k\rangle = \sqrt{[k+1]_q}|k+1\rangle$$
$$a|k\rangle = \sqrt{[k]_q}|k-1\rangle.$$

A different q-exponential is discussed by Ismail and Zhang [2], and different q-bosons are introduced in [3–5]. Several attempts to consider the interrelations among the various types of boson operators have been presented, cf [6]. Within the stricter framework of quantum

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groups the Hopf algebra structure provides a further rather powerful guideline, which we shall not consider in the present paper.

Defining the number operator $\hat{n}$ via

$$\hat{n}|k\rangle = k|k\rangle$$

we obtain

$$[a, \hat{n}] = a \qquad [\hat{n}, a^\dagger] = a^\dagger. \tag{2}$$

Furthermore

$$a^\dagger a|k\rangle = [k]_q|k\rangle \qquad \text{i.e. } [\hat{n}]_q = a^\dagger a.$$

Hence

$$[a, [\hat{n}]_q]_q = a \qquad [[\hat{n}]_q, a^\dagger]_q = a^\dagger \tag{3}$$

and in the limit $q \to 1$ equations (2) and (3) coincide. For $q \neq 1$ both are viable q-deformations of the corresponding boson commutation relations.

The transformation of a second-quantized expression into a normally ordered form, in which each term is written with the creation operators preceding the annihilation operators, has been found to simplify quantum mechanical calculations in a large and varied range of situations. Techniques for the accomplishment of this ordering have been developed extensively, and are widely utilized. A particular subclass of problems and techniques involves situations in which the operators of interest commute with the number operator. More specifically, one is interested in transforming an operator which is a function of the number operator into a normally ordered form, or transforming an operator which is a sum of terms, each one of which consists of an equal number of creation and annihilation operators, into an equivalent operator expressed in terms of the number operator only. The coefficients in the expression for an integral power of the boson number operator as a normally-ordered polynomial in the creation and annihilation operators turn out to be Stirling numbers of the second kind [7].

The normal ordering of powers of the number operator for Arik–Coon q-bosons exhibits a rather minor deviation from the corresponding result for conventional bosons. The expansion coefficients were identified as q-Stirling numbers of the second kind [8], which were introduced in the context of q-analysis a long time ago, and whose combinatorial significance has been extensively studied [9–15].

The normally ordered expansion of a power of the number operator for deformed bosons other than the Arik–Coon q-bosons considered here differs in a significant respect from that for conventional bosons. It is found that the coefficients, that generalize the Stirling (or q-Stirling) numbers, depend on the operator $\hat{n}$ [8]. In view of this marked distinction between Arik–Coon q-bosons and all others, we restrict our attention in the present paper to the former, in terms of which we consider the q-analogues of a family of Stirling number identities. Similar reordering problems, that involve the operator relation $AB - qBA = B$, have been discussed by Al-Salam and Ismail [16].

2. The q-binomial theorem

For two commuting variables x and y the binomial theorem states that

$$(x + y)^k = \sum_{\ell=0}^{k} \binom{k}{\ell} x^\ell y^{k-\ell}. \tag{4}$$

Since $\binom{k}{\ell} = k!/\ell!(k-\ell)!$ it is natural to define a q-analogue of the binomial coefficient in terms of the q-analogue of the factorial, introduced earlier, thus, $\binom{k}{\ell}_q = [k]_q!/[\ell]_q![k-\ell]_q!$. It turns out that this deformation satisfies the identity [17]

$$(x+1)(x+q)(x+q^2)\cdots(x+q^{k-1}) = \sum_{\ell=0}^{k} \binom{k}{\ell}_q x^{k-\ell} q^{\ell(\ell-1)/2} \tag{5}$$

which is easily proved by induction over k, using the well known q-binomial recursion

$$\binom{k+1}{\ell}_q = \binom{k}{\ell-1}_q q^{k+1-\ell} + \binom{k}{\ell}_q = \binom{k}{\ell-1}_q + \binom{k}{\ell}_q q^{\ell}$$

that is the basis of the q-Pascal triangle. The last identity follows from the definition of the q-binomial coefficients by utilizing equation (1). The same q-analogue of the binomial coefficients appears in the expression

$$(x+y)^k = \sum_{\ell=0}^{k} \binom{k}{\ell}_q x^{\ell} y^{k-\ell}$$

for two variables that satisfy $yx = qxy$ [18]. In fact, for such variables it is a simple matter to express $(x+y)^k$ in the form

$$\begin{aligned}(x+y)^k &= x^k(1+q^{k-1}x^{-1}y)(1+q^{k-2}x^{-1}y)\cdots(1+qx^{-1}y)(1+x^{-1}y)\\ &= q^{-(k-1)k/2}y^k(z+1)(z+q)\cdots(z+q^{k-1})\end{aligned}$$

where $z = y^{-1}x$. Hence, using (5), we have

$$(x+y)^k = q^{-(k-1)k/2}y^k \sum_{\ell=0}^{k} \binom{k}{\ell}_q z^{k-\ell} q^{(\ell-1)\ell/2} = \sum_{\ell=0}^{k} \binom{k}{\ell}_q x^{k-\ell} y^{\ell}.$$

3. Stirling numbers: basic properties

Our presentation of Stirling numbers and of their properties follows Graham *et al* [19], to whose notation and phase conventions we adhere.

Stirling numbers of the second kind are defined as the transformation coefficients from the falling powers, $x^{\underline{k}} \equiv x(x-1)(x-2)\cdots(x-k+1)$, to the powers

$$x^k = \sum_{\ell=1}^{k} \left\{ {k \atop \ell} \right\} x^{\underline{\ell}}.$$

Stirling numbers of the first kind are defined via

$$x^{\overline{k}} = \sum_{\ell=1}^{k} \left[{k \atop \ell} \right] x^{\ell}$$

where $x^{\overline{k}} = x(x+1)(x+2)\cdots(x+k-1)$, or via

$$x^{\underline{k}} = \sum_{\ell=1}^{k} \left[{k \atop \ell} \right] (-1)^{k-\ell} x^{\ell}. \tag{6}$$

The equivalence of these two definitions follows from the obvious relation $x^{\underline{k}} = (-1)^k(-x)^{\overline{k}}$. These defining relations should be supplemented by

$$\left[{k \atop 0} \right] = \left\{ {k \atop 0} \right\} = \delta_{k,0}. \tag{7}$$

From the defining relations one easily obtains the basic properties, i.e. the recurrence relations

$$\genfrac{\{}{\}}{0pt}{}{k+1}{\ell} = \genfrac{\{}{\}}{0pt}{}{k}{\ell-1} + \ell \genfrac{\{}{\}}{0pt}{}{k}{\ell} \tag{8}$$

and

$$\genfrac{[}{]}{0pt}{}{k+1}{\ell} = \genfrac{[}{]}{0pt}{}{k}{\ell-1} + k \genfrac{[}{]}{0pt}{}{k}{\ell} \tag{9}$$

and the inversion formulae

$$\sum_{\ell=m}^{k} \genfrac{[}{]}{0pt}{}{k}{\ell} \genfrac{\{}{\}}{0pt}{}{\ell}{m} (-1)^{k-\ell} = \delta_{k,m} \tag{10}$$

and

$$\sum_{\ell=m}^{k} \genfrac{\{}{\}}{0pt}{}{k}{\ell} \genfrac{[}{]}{0pt}{}{\ell}{m} (-1)^{k-\ell} = \delta_{k,m}. \tag{11}$$

The connection with the normal ordering problem of boson operators is as follows [7]

$$(a^\dagger)^k a^k = \hat{n}^{\underline{k}} = \sum_{\ell=1}^{k} \genfrac{[}{]}{0pt}{}{k}{\ell} (-1)^{k-\ell} \hat{n}^\ell$$

and

$$\hat{n}^k = \sum_{\ell=1}^{k} \genfrac{\{}{\}}{0pt}{}{k}{\ell} \hat{n}^{\underline{\ell}} = \sum_{\ell=1}^{k} \genfrac{\{}{\}}{0pt}{}{k}{\ell} (a^\dagger)^\ell a^\ell.$$

The q-analogues of these relations can be written in terms of Arik–Coon q-bosons, that satisfy the q-commutation relation $[a, a^\dagger]_q = 1$. The defining relations for q-Stirling numbers can be written in the form [8]

$$(a^\dagger)^k a^k = \sum_{\ell=1}^{k} \genfrac{[}{]}{0pt}{}{k}{\ell}_q (-1)^{k-\ell} [\hat{n}]_q^\ell$$

and

$$[\hat{n}]_q^k = \sum_{\ell=1}^{k} \genfrac{\{}{\}}{0pt}{}{k}{\ell}_q (a^\dagger)^\ell a^\ell$$

where equation (7) remains unchanged. The q-Stirling numbers satisfy the recurrence relations

$$\genfrac{\{}{\}}{0pt}{}{k+1}{\ell}_q = \genfrac{\{}{\}}{0pt}{}{k}{\ell-1}_q q^{\ell-1} + [\ell]_q \genfrac{\{}{\}}{0pt}{}{k}{\ell}_q \tag{12}$$

and

$$\genfrac{[}{]}{0pt}{}{k+1}{\ell}_q = \left(\genfrac{[}{]}{0pt}{}{k}{\ell-1}_q + [k]_q \genfrac{[}{]}{0pt}{}{k}{\ell}_q \right) q^{-k} \tag{13}$$

and the inversion formulae

$$\sum_{\ell=m}^{k} \genfrac{[}{]}{0pt}{}{k}{\ell}_q \genfrac{\{}{\}}{0pt}{}{\ell}{m}_q (-1)^{k-\ell} = \delta_{k,m} \tag{14}$$

and

$$\sum_{\ell=m}^{k} \left\{ {k \atop \ell} \right\}_q \left[{\ell \atop m} \right]_q (-1)^{k-\ell} = \delta_{k,m}. \tag{15}$$

The q-Stirling numbers were originally introduced in terms of q-falling powers [9]

$$[x]_q^{\underline{k}} = [x]_q [x-1]_q \cdots [x-k+1]_q$$

via the relations

$$[x]_q^{\underline{k}} = \sum_{\ell=1}^{k} \left[{k \atop \ell} \right]_q (-1)^{k-\ell} [x]_q^{\ell}$$

and

$$[x]_q^{k} = \sum_{\ell=1}^{k} \left\{ {k \atop \ell} \right\}_q [x]_q^{\underline{\ell}}.$$

The recurrence relations and inversion formulae satisfied by the Stirling and q-Stirling numbers can easily be derived (*ab initio*) by considering the transformations between the two sets of operators $\{a^\dagger a, (a^\dagger)^2 a^2, \ldots, (a^\dagger)^k a^k\}$ and $\{\hat{n}, \hat{n}^2, \ldots, \hat{n}^k\}$, as was actually done in [8].

4. Stirling number identities and their q-analogues

We now consider q-analogues of the following Stirling number identities, listed by Graham *et al* [19]. A derivation is presented for each one of them, using the boson operator algebra $[a, a^\dagger]_q = 1$ or the q-commuting coordinates $[y, x]_q = 0$, as far as possible. q-analogues are obtained by introducing appropriate modifications in the derivations. It is certainly conceivable that different routes could lead to different q-analogues. In some of the cases considered this is explicitly pointed out.

Identity 1.

$$\left\{ {k+1 \atop m+1} \right\} = \sum_{\ell=m}^{k} \binom{k}{\ell} \left\{ {\ell \atop m} \right\}.$$

Proof. Noting the identity

$$a^\dagger (\hat{n}+1)^k a = a^\dagger a (\hat{n})^k = \hat{n}^{k+1} \tag{16}$$

we evaluate the left-hand side as

$$a^\dagger (\hat{n}+1)^k a = a^\dagger \left(\sum_{\ell=0}^{k} \binom{k}{\ell} \hat{n}^{\ell} \right) a = \sum_{\ell=0}^{k} \binom{k}{\ell} \sum_{m=0}^{\ell} \left\{ {\ell \atop m} \right\} (a^\dagger)^{m+1} a^{m+1}$$

and the right-hand side as

$$\hat{n}^{k+1} = \sum_{p=1}^{k+1} \left\{ {k+1 \atop p} \right\} (a^\dagger)^p a^p.$$

Equating coefficients we obtain identity 1. □

q-analogue. Relation (16) has a q-analogue of the form $a^\dagger[\hat{n}+1]_q^k a = [\hat{n}]_q^{k+1}$. Noting that $[\hat{n}+1]_q = 1 + q[\hat{n}]$ we evaluate the left-hand side as

$$a^\dagger[\hat{n}+1]_q^k a = a^\dagger \left(\sum_{\ell=0}^{k} \binom{k}{\ell} q^\ell [\hat{n}]_q^\ell \right) a$$

and proceed in complete analogy with the derivation presented to obtain

$$\left\{ \begin{matrix} k+1 \\ m+1 \end{matrix} \right\}_q = \sum_{\ell=m}^{k} \binom{k}{\ell} \left\{ \begin{matrix} \ell \\ m \end{matrix} \right\}_q q^\ell.$$

Note that this relation involves the q-analogue of Stirling numbers of the second kind, but the undeformed binomial coefficients. It is intriguing to inquire whether an identity that involves the q-analogue of the binomial coefficients can be formulated.

Identity 2.

$$\left[\begin{matrix} k+1 \\ m+1 \end{matrix} \right] = \sum_{\ell=m}^{k} \left[\begin{matrix} k \\ \ell \end{matrix} \right] \binom{\ell}{m}.$$

Proof. We evaluate $(a^\dagger)^{k+1} a^{k+1}$ in two different ways. On the one hand

$$(a^\dagger)^{k+1} a^{k+1} = \sum_{\ell=1}^{k+1} \left[\begin{matrix} k+1 \\ \ell \end{matrix} \right] (-1)^{k+1-\ell} \hat{n}^\ell$$

and on the other hand

$$\begin{aligned} (a^\dagger)^{k+1} a^{k+1} &= a^\dagger \left(\sum_{\ell=1}^{k} \left[\begin{matrix} k \\ \ell \end{matrix} \right] (-1)^{k-\ell} \hat{n}^\ell \right) a \\ &= \sum_{\ell=1}^{k} \left[\begin{matrix} k \\ \ell \end{matrix} \right] (-1)^{k-\ell} (\hat{n}-1)^\ell \hat{n} = \sum_{\ell=1}^{k} \sum_{m=0}^{\ell} \left[\begin{matrix} k \\ \ell \end{matrix} \right] \binom{\ell}{m} (-1)^{k-m} \hat{n}^{m+1}. \end{aligned}$$

Identity 2 follows by equating coefficients of equal powers of $\hat{n}$. □

q-analogue. The derivation follows the procedure described earlier with obvious minor adjustments. One obtains

$$\left[\begin{matrix} k+1 \\ m+1 \end{matrix} \right]_q = \sum_{\ell=m}^{k} \left[\begin{matrix} k \\ \ell \end{matrix} \right]_q \binom{\ell}{m} q^{-\ell}.$$

Again, the binomial coefficient is not deformed. A combinatorial derivation of this identity was presented by de Médicis and Leroux [13].

Identity 3.

$$\left\{ \begin{matrix} k \\ m \end{matrix} \right\} = \sum_{\ell=m}^{k} \binom{k}{\ell} \left\{ \begin{matrix} \ell+1 \\ m+1 \end{matrix} \right\} (-1)^{k-\ell}.$$

Proof. We evaluate $a^\dagger \hat{n}^k a$ in two different ways, first

$$a^\dagger \hat{n}^k a = \sum_{\ell=1}^{k} \left\{ {k \atop \ell} \right\} (a^\dagger)^{\ell+1} a^{\ell+1}$$

and second

$$\begin{aligned} a^\dagger \hat{n}^k a = \hat{n}(\hat{n}-1)^k &= \sum_{m=0}^{k} \binom{k}{m} \hat{n}^{m+1} (-1)^{k-m} \\ &= \sum_{m=0}^{k} \binom{k}{m} (-1)^{k-m} \sum_{p=1}^{m+1} \left\{ {m+1 \atop p} \right\} (a^\dagger)^p a^p. \end{aligned}$$

Equating coefficients we obtain identity 3. □

q-analogue. We note that on the one hand

$$a^\dagger [\hat{n}]_q^k a = \sum_{\ell=1}^{k} \left\{ {k \atop \ell} \right\}_q (a^\dagger)^{\ell+1} a^{\ell+1}$$

and on the other hand

$$\begin{aligned} a^\dagger [\hat{n}]_q^k a = [\hat{n}]_q [\hat{n}-1]_q^k = [\hat{n}]_q \left(\frac{1}{q} ([\hat{n}]_q - 1) \right)^k \\ = \frac{1}{q^k} \sum_{\ell=0}^{k} \binom{k}{\ell} (-1)^{k-m} \sum_{p=1}^{\ell+1} \left\{ {\ell+1 \atop p} \right\}_q (a^\dagger)^p a^p. \end{aligned}$$

Thereby equating coefficients we obtain

$$\left\{ {k \atop m} \right\}_q = \frac{1}{q^k} \sum_{\ell=m}^{k} \binom{k}{\ell} \left\{ {\ell+1 \atop m+1} \right\}_q (-1)^{k-\ell}.$$

Identity 4.

$$\left[{k \atop m} \right] = \sum_{\ell=m}^{k} \left[{k+1 \atop \ell+1} \right] \binom{\ell}{m} (-1)^{\ell-m}.$$

Proof. We evaluate $(a^\dagger)^{k+1} a^{k+1}$ in two ways, first

$$\begin{aligned} (a^\dagger)^{k+1} a^{k+1} = a^\dagger ((a^\dagger)^k a^k) a = a^\dagger \left(\sum_{\ell=1}^{k} \left[{k \atop \ell} \right] (-1)^{k-\ell} \hat{n}^\ell \right) a \\ = \sum_{\ell=1}^{k} \left[{k \atop \ell} \right] (-1)^{k-\ell} \hat{n} (\hat{n}-1)^\ell \end{aligned}$$

and second

$$\begin{aligned} (a^\dagger)^{k+1} a^{k+1} &= \sum_{p=1}^{k+1} \left[{k+1 \atop p} \right] (-1)^{k+1-p} \hat{n}^p \\ &= \sum_{p=1}^{k+1} \left[{k+1 \atop p} \right] (-1)^{k+1-p} \hat{n} (1 + (\hat{n}-1))^{p-1} \\ &= \sum_{p=1}^{k+1} \sum_{\ell=0}^{p-1} \left[{k+1 \atop p} \right] \binom{p-1}{\ell} (-1)^{k+1-p} \hat{n} (\hat{n}-1)^\ell \end{aligned}$$

and then, equating coefficients of equal powers of $(\hat{n}-1)$, we obtain identity 4. □

q-analogue. The derivation presented above proceeds virtually unchanged, except that the substitution $\hat{n} = 1 + (\hat{n} - 1)$ should be replaced by $[\hat{n}]_q = 1 + q[\hat{n} - 1]_q$. One finally obtains

$$\begin{bmatrix} k \\ m \end{bmatrix}_q = \sum_{\ell=m}^{k} \begin{bmatrix} k+1 \\ \ell+1 \end{bmatrix}_q \binom{\ell}{m} q^m (-1)^{\ell-m}$$

and once more, the binomial coefficient remains undeformed.

Identity 5.

$$m! \begin{Bmatrix} k \\ m \end{Bmatrix} = \sum_{\ell=0}^{m} \binom{m}{\ell} \ell^k (-1)^{m-\ell}.$$

Proof. We proceed by induction over k. First, we check that the identity holds for $k = 1$, noting that both sides vanish for $m = 0$ as well as for $m > 1$, and that for $m = 1$ both sides are equal to 1.

Assuming that the identity holds for k (and all m) we prove that it holds for $k+1$. This is done as follows: we write the identity for $(k, m-1)$ and for (k, m), i.e.

$$(m-1)! \begin{Bmatrix} k \\ m-1 \end{Bmatrix} = \sum_{\ell=0}^{m-1} \binom{m-1}{\ell} \ell^k (-1)^{m-1-\ell}$$

and

$$m! \begin{Bmatrix} k \\ m \end{Bmatrix} = \sum_{\ell=0}^{m} \binom{m}{\ell} \ell^k (-1)^{m-\ell}.$$

Adding these two identities we obtain

$$(m-1)! \left(\begin{Bmatrix} k \\ m-1 \end{Bmatrix} + m \begin{Bmatrix} k \\ m \end{Bmatrix} \right) = \sum_{\ell=0}^{m} \left(\binom{m}{\ell} - \binom{m-1}{\ell} \right) \ell^k (-1)^{m-\ell}.$$

Using the recurrence relation for Stirling numbers of the first kind (9), and the binomial recursion $\binom{m}{\ell} - \binom{m-1}{\ell} = \binom{m-1}{\ell-1}$, and multiplying both the left- and the right-hand sides by m we complete the proof. $\square$

q-analogue. The identity

$$[m]_q! \begin{Bmatrix} k \\ m \end{Bmatrix}_q = \sum_{\ell=0}^{m} \binom{m}{\ell}_q [\ell]_q^k (-1)^{m-\ell} q^{(m-\ell)(m-\ell-1)/2}$$

can be derived by an induction procedure that follows very closely the procedure presented.

Identity 6.

$$\begin{Bmatrix} k+1 \\ m+1 \end{Bmatrix} = \sum_{\ell=m}^{k} \begin{Bmatrix} \ell \\ m \end{Bmatrix} (m+1)^{k-\ell}.$$

Proof. This identity follows by induction over k and using the recurrence relation for Stirling numbers of the second kind (8). $\square$

q-analogue. Induction over k establishes the identity

$$\begin{Bmatrix} k+1 \\ m+1 \end{Bmatrix}_q = q^m \sum_{\ell=m}^{k} \begin{Bmatrix} \ell \\ m \end{Bmatrix}_q [m+1]_q^{k-\ell}.$$

Identity 7.

$$\begin{bmatrix} k+1 \\ m+1 \end{bmatrix} = \sum_{\ell=m}^{k} \begin{bmatrix} \ell \\ m \end{bmatrix} \frac{k!}{\ell!}.$$

Proof. By induction over k, starting from $k = m$ and using the recurrence relation for Stirling numbers of the first kind, (9), the identity follows. □

q-analogue. The identity

$$\begin{bmatrix} k+1 \\ m+1 \end{bmatrix}_q = \sum_{\ell=m}^{k} \begin{bmatrix} \ell \\ m \end{bmatrix}_q \frac{[k]_q!}{[\ell]_q!} q^{-(k+1-\ell)(k+\ell)/2}$$

is established by induction over k.

Identity 8.

$$\left\{ \begin{matrix} m+k+1 \\ m \end{matrix} \right\} = \sum_{\ell=0}^{m} \ell \left\{ \begin{matrix} k+\ell \\ \ell \end{matrix} \right\}.$$

Proof. This identity follows by repeated application of the recurrence relation, equation (8). □

q-analogue. Repeated application of the recurrence relation, equation (12), yields

$$\left\{ \begin{matrix} m+k+1 \\ m \end{matrix} \right\}_q = \sum_{\ell=0}^{m} [\ell]_q \left\{ \begin{matrix} k+\ell \\ \ell \end{matrix} \right\}_q q^{(m+\ell-1)(m-\ell)/2}.$$

Identity 9.

$$\begin{bmatrix} m+k+1 \\ m \end{bmatrix} = \sum_{\ell=0}^{m} (k+\ell) \begin{bmatrix} k+\ell \\ \ell \end{bmatrix}.$$

Proof. This identity follows by repeated application of the recurrence relation, equation (9). □

q-analogue. Repeated application of the recurrence relation (13) yields

$$\begin{bmatrix} m+k+1 \\ m \end{bmatrix}_q = \sum_{\ell=0}^{m} q^{-(2k+m+\ell)(m-\ell+1)/2} [k+\ell]_q \begin{bmatrix} k+\ell \\ \ell \end{bmatrix}_q.$$

Identity 10.

$$\binom{k}{m} = \sum_{\ell=m}^{k} \left\{ \begin{matrix} k+1 \\ \ell+1 \end{matrix} \right\} \begin{bmatrix} \ell \\ m \end{bmatrix} (-1)^{\ell-m}.$$

Proof. Substituting the expression for $\left\{ \begin{smallmatrix} k+1 \\ m+1 \end{smallmatrix} \right\}$ from identity 1 in the left-hand side and using the inversion formula, equation (11), we obtain the right-hand side. □

q-analogue. Following the same procedure we obtain

$$\binom{k}{m} q^m = \sum_{\ell=m}^{k} \left\{ \begin{matrix} k+1 \\ \ell+1 \end{matrix} \right\}_q \left[\begin{matrix} \ell \\ m \end{matrix} \right]_q (-1)^{\ell-m}$$

and note that the binomial coefficient on the left-hand side is not deformed.

Lemma 1.

$$(x-1)^{\underline{k}} = k! \sum_{m=0}^{k} (-1)^{k-m} \frac{x^{\underline{m}}}{m!}.$$

Proof. By induction over k. □

q-analogue.

$$[x-1]_q^{\underline{k}} = [k]_q! \sum_{m=0}^{k} (-1)^{k-m} q^{(m(m-1)-k(k+1))/2} \frac{[x]_q^{\underline{m}}}{[m]_q!}.$$

Proof. By induction over k. □

Identity 11.

$$\frac{k!}{m!} = \sum_{\ell} \left[\begin{matrix} k+1 \\ \ell+1 \end{matrix} \right] \left\{ \begin{matrix} \ell \\ m \end{matrix} \right\} (-1)^{m-\ell}.$$

Proof. We write $(x-1)^{\underline{m}}$ in two different ways. From the defining relation of Stirling numbers of the first kind (6) that we write for $x^{\underline{m+1}}$, we obtain after dividing both sides by x

$$\begin{aligned} (x-1)^{\underline{m}} &= \sum_{\ell=1}^{m+1} \left[\begin{matrix} m+1 \\ \ell \end{matrix} \right] (-1)^{m+1-\ell} x^{\ell-1} \\ &= \sum_{\ell=1}^{m+1} \sum_{p=1}^{\ell-1} \left[\begin{matrix} m+1 \\ \ell \end{matrix} \right] \left\{ \begin{matrix} \ell-1 \\ p \end{matrix} \right\} (-1)^{m+1-\ell} x^{\underline{p}}. \end{aligned}$$

From lemma 1 we obtain

$$(x-1)^{\underline{m}} = \sum_{\ell=0}^{m} (-1)^{m-\ell} \frac{k!}{\ell!} x^{\underline{\ell}}$$

and by equating the coefficients of $x^{\underline{\ell}}$ we obtain identity 11. □

q-analogue. In complete analogy we derive the identity

$$\frac{[k]_q!}{[m]_q!} q^{(m(m-1)-k(k+1))/2} = \sum_{\ell=m}^{k} \left[\begin{matrix} k+1 \\ \ell+1 \end{matrix} \right]_q \left\{ \begin{matrix} \ell \\ m \end{matrix} \right\}_q (-1)^{\ell-m}.$$

Lemma 2. (*Vandermonde's formula* [20]). For x and y two commuting variables the following relation holds

$$(x+y)^{\underline{k}} = \sum_{\ell=0}^{k} \binom{k}{\ell} x^{\underline{\ell}} y^{\underline{k-\ell}}. \tag{17}$$

Proof. By induction over k, first checking that the relation holds for $k = 1$ and then noting that

$$
\begin{aligned}
(x+y)^{\underline{k+1}} &= (x+y)^{\underline{k}}(x+y-k) \\
&= \sum_{\ell=0}^{k} \binom{k}{\ell} x^{\underline{k}} y^{\underline{k-\ell}}((x-\ell)+(y-(k-\ell)))
\end{aligned}
$$

easily yields the desired result. For $y = -1$ lemma 2 reduces to lemma 1. $\square$

q-analogue. For two variables x and y that satisfy $yx = qxy$ we express $(x+y)^{\underline{k}}$ in terms of a linear combination of terms of the form $x^{\underline{r}}y^{\underline{s}}$ in the following way

$$
\begin{aligned}
(x+y)^{\underline{k}} &= \sum_{\ell=1}^{k} \left[{k \atop \ell} \right] (-1)^{k-\ell} \sum_{m=0}^{\ell} \binom{\ell}{m}_q \left(\sum_{r=1}^{m} \left\{ {m \atop r} \right\} x^{\underline{r}} \right) \left(\sum_{s=1}^{\ell-m} \left\{ {\ell-m \atop s} \right\} y^{\underline{s}} \right) \\
&= \sum_r \sum_s \left(\!\!\left(\begin{matrix} & k & \\ r & & s \end{matrix} \right)\!\!\right)_q x^{\underline{r}} y^{\underline{s}}
\end{aligned}
\tag{18}
$$

i.e.

$$
\left(\!\!\left(\begin{matrix} & k & \\ r & & s \end{matrix} \right)\!\!\right)_q = \sum_\ell \sum_m \left[{k \atop \ell} \right] (-1)^{k-\ell} \binom{\ell}{m}_q \left\{ {m \atop r} \right\} \left\{ {\ell-m \atop s} \right\} \qquad (r+s \leqslant k).
$$

Note that here the binomial coefficient is deformed but the Stirling numbers are not. In view of its role in equation (18) we shall refer to $\left(\!\left({k \atop rs} \right)\!\right)_q$ as the q-falling binomial coefficient. It is easy to establish that for $q = 1$

$$
\left(\!\!\left(\begin{matrix} & k & \\ r & & s \end{matrix} \right)\!\!\right) = \binom{k}{r} \delta_{k,r+s}.
$$

As an illustration of the q-falling binomial coefficients we note that

$$
\begin{aligned}
(x+y)^{\underline{1}} &= x^{\underline{1}} + y^{\underline{1}} \\
(x+y)^{\underline{2}} &= x^{\underline{2}} + (1+q)x^{\underline{1}}y^{\underline{1}} + y^{\underline{2}} \\
(x+y)^{\underline{3}} &= x^{\underline{3}} + (1+q+q^2)x^{\underline{2}}y^{\underline{1}} + (1+q+q^2)x^{\underline{1}}y^{\underline{2}} + y^{\underline{3}} + (2q^2-q-1)x^{\underline{1}}y^{\underline{1}}
\end{aligned}
$$

i.e.

$$
\begin{aligned}
&\left(\!\!\left(\begin{matrix} & 1 & \\ 1 & & 0 \end{matrix} \right)\!\!\right)_q = 1 \qquad \left(\!\!\left(\begin{matrix} & 1 & \\ 0 & & 1 \end{matrix} \right)\!\!\right)_q = 1 \\
&\left(\!\!\left(\begin{matrix} & 2 & \\ 2 & & 0 \end{matrix} \right)\!\!\right)_q = 1 \qquad \left(\!\!\left(\begin{matrix} & 2 & \\ 1 & & 1 \end{matrix} \right)\!\!\right)_q = [2]_q \qquad \left(\!\!\left(\begin{matrix} & 2 & \\ 0 & & 2 \end{matrix} \right)\!\!\right)_q = 1 \\
&\left(\!\!\left(\begin{matrix} & 3 & \\ 3 & & 0 \end{matrix} \right)\!\!\right)_q = 1 \qquad \left(\!\!\left(\begin{matrix} & 3 & \\ 2 & & 1 \end{matrix} \right)\!\!\right)_q = [3]_q \qquad \left(\!\!\left(\begin{matrix} & 3 & \\ 1 & & 2 \end{matrix} \right)\!\!\right)_q = [3]_q \\
&\left(\!\!\left(\begin{matrix} & 3 & \\ 0 & & 3 \end{matrix} \right)\!\!\right)_q = 1 \qquad \left(\!\!\left(\begin{matrix} & 3 & \\ 1 & & 1 \end{matrix} \right)\!\!\right)_q = 2q^2 - q - 1.
\end{aligned}
$$

Thus, while the binomial coefficients appear both in the binomial theorem (4) and in Vandermonde's formula (17) the q-analogues of these theorems give rise to two distinct sets of q-analogues of the binomial coefficients.

Identity 12.

$$
\left\{ {n \atop \ell+m} \right\} \binom{\ell+m}{\ell} = \sum_k \left\{ {k \atop \ell} \right\} \left\{ {n-k \atop m} \right\} \binom{n}{k}.
$$

Proof. We write $(x+y)^n$ in two different ways. On the one hand

$$(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} \sum_{r=1}^{k} \left\{ \begin{matrix} k \\ r \end{matrix} \right\} x^{\underline{r}} \sum_{s=1}^{n-k} \left\{ \begin{matrix} n-k \\ s \end{matrix} \right\} y^{\underline{s}}$$

and on the other hand

$$(x+y)^n = \sum_{p=1}^{n} \left\{ \begin{matrix} n \\ p \end{matrix} \right\} \sum_{u=0}^{p} \binom{p}{u} x^{\underline{u}}\, y^{\underline{p-u}}.$$

Equating coefficients of $x^{\underline{r}} y^{\underline{s}}$ we obtain identity 12. □

q-analogue. Let the two variables x and y satisfy $yx = qxy$. $(x+y)^n$ can be written in two different ways. We present the steps of the derivation in order to carefully distinguish between undeformed and deformed quantities that appear in the various steps. First, we have

$$(x+y)^n = \sum_{k=0}^{n} \binom{n}{k}_q x^k y^{n-k} = \sum_{k=0}^{n} \binom{n}{k}_q \sum_{\ell=1}^{k} \left\{ \begin{matrix} k \\ \ell \end{matrix} \right\} x^{\underline{\ell}} \sum_{m=1}^{n-k} \left\{ \begin{matrix} n-k \\ m \end{matrix} \right\} y^{\underline{m}}$$

and second, using lemma 2 we have

$$(x+y)^n = \sum_{p=1}^{n} \left\{ \begin{matrix} n \\ p \end{matrix} \right\} (x+y)^{\underline{p}} = \sum_{p=1}^{n} \left\{ \begin{matrix} n \\ p \end{matrix} \right\} \sum_{r} \sum_{s} \left(\left(\begin{matrix} & p & \\ r & & s \end{matrix} \right)\right)_q x^{\underline{r}} y^{\underline{s}}.$$

Equating coefficients we obtain

$$\sum_{p} \left\{ \begin{matrix} n \\ p \end{matrix} \right\} \left(\left(\begin{matrix} & p & \\ r & & s \end{matrix} \right)\right)_q = \sum_{k} \left\{ \begin{matrix} k \\ r \end{matrix} \right\} \left\{ \begin{matrix} n-k \\ s \end{matrix} \right\} \binom{n}{k}_q .$$

Identity 13.

$$\begin{bmatrix} n \\ \ell+m \end{bmatrix} \binom{\ell+m}{\ell} = \sum_{k} \binom{n}{k} \begin{bmatrix} k \\ \ell \end{bmatrix} \begin{bmatrix} n-k \\ m \end{bmatrix}.$$

Proof. For the two commuting variables x and y we write $(x+y)^{\underline{n}}$ in two different ways. On the one hand

$$(x+y)^{\underline{n}} = \sum_{k=0}^{n} \binom{n}{k} \left(\sum_{\ell=0}^{k} \begin{bmatrix} k \\ \ell \end{bmatrix} (-1)^{k-\ell} x^{\ell} \right) \left(\sum_{m=0}^{n-k} \begin{bmatrix} n-k \\ m \end{bmatrix} (-1)^{n-k-m} y^{m} \right)$$

and on the other hand

$$(x+y)^{\underline{n}} = \sum_{p=1}^{n} \begin{bmatrix} n \\ p \end{bmatrix} (-1)^{n-p} \sum_{t=0}^{p} \binom{p}{t} x^t y^{p-t}.$$

Equating coefficients we obtain identity 13. □

q-analogue. For x and y satisfying $yx = qxy$ we write

$$\begin{aligned} (x+y)^{\underline{n}} &= \sum_{r,s} \left(\left(\begin{matrix} & n & \\ r & & s \end{matrix} \right)\right)_q x^{\underline{r}} y^{\underline{s}} \\ &= \sum_{r,s} \sum_{\ell} \sum_{m} \left(\left(\begin{matrix} & p & \\ r & & s \end{matrix} \right)\right)_q \begin{bmatrix} r \\ \ell \end{bmatrix} \begin{bmatrix} s \\ m \end{bmatrix} (-1)^{r+s-\ell-m} x^{\ell} y^{m} \end{aligned}$$

and then

$$(x+y)^{\underline{n}} = \sum_p \begin{bmatrix} n \\ p \end{bmatrix} (-1)^{n-p}(x+y)^p = \sum_p \sum_t \begin{bmatrix} n \\ p \end{bmatrix} (-1)^{n-p} \binom{p}{t}_q x^t y^{p-t}.$$

Equating coefficients we obtain

$$\begin{bmatrix} n \\ \ell+m \end{bmatrix} \binom{\ell+m}{\ell}_q = \sum_{r,s} \left(\left(\begin{matrix} & p & \\ r & & s \end{matrix}\right)\right)_q \begin{bmatrix} r \\ \ell \end{bmatrix} \begin{bmatrix} s \\ m \end{bmatrix} (-1)^{n-r-s}.$$

Note that the binomial coefficients on the left- and right-hand sides are deformed in two different ways; the Stirling numbers remain undeformed.

Two further identities presented by Graham *et al* [19] relate the Stirling numbers of the two kinds to one another. Their derivation is presented by Jordan [21], but their q-analogues have so far evaded our efforts. They are therefore left as a challenge to the reader.

5. Conclusions

The q-analogues of a family of identities that involve Stirling numbers of the first and second kinds have been derived. The derivation takes maximum advantage of the connection between the q-Stirling numbers and the normal ordering problem for Arik–Coon type q-bosons.

It is remarkable that q-analogues of many of the classical Stirling number identities can be formulated and we have revealed several of their interesting features. Among others these include the emergence of a new type of deformed binomial coefficient, that, due to the role it plays, was referred to as the q-falling binomial coefficient.

One issue of particular interest has to do with the fact that some of the Stirling number identities examined have q-analogues in which some but not all the factors involved are q-deformed. The possibility that other routes could lead to other types of q-analogues in which different factors, perhaps all of them, would be q-deformed, presents an interesting set of open problems.

Whereas further generalization of at least some of the results to other types of q-bosons is conceivable, it is clear that the deviation from the undeformed bosons will be considerably more far reaching.

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